A Predator-Prey Model:

The Lotka-Volterra System

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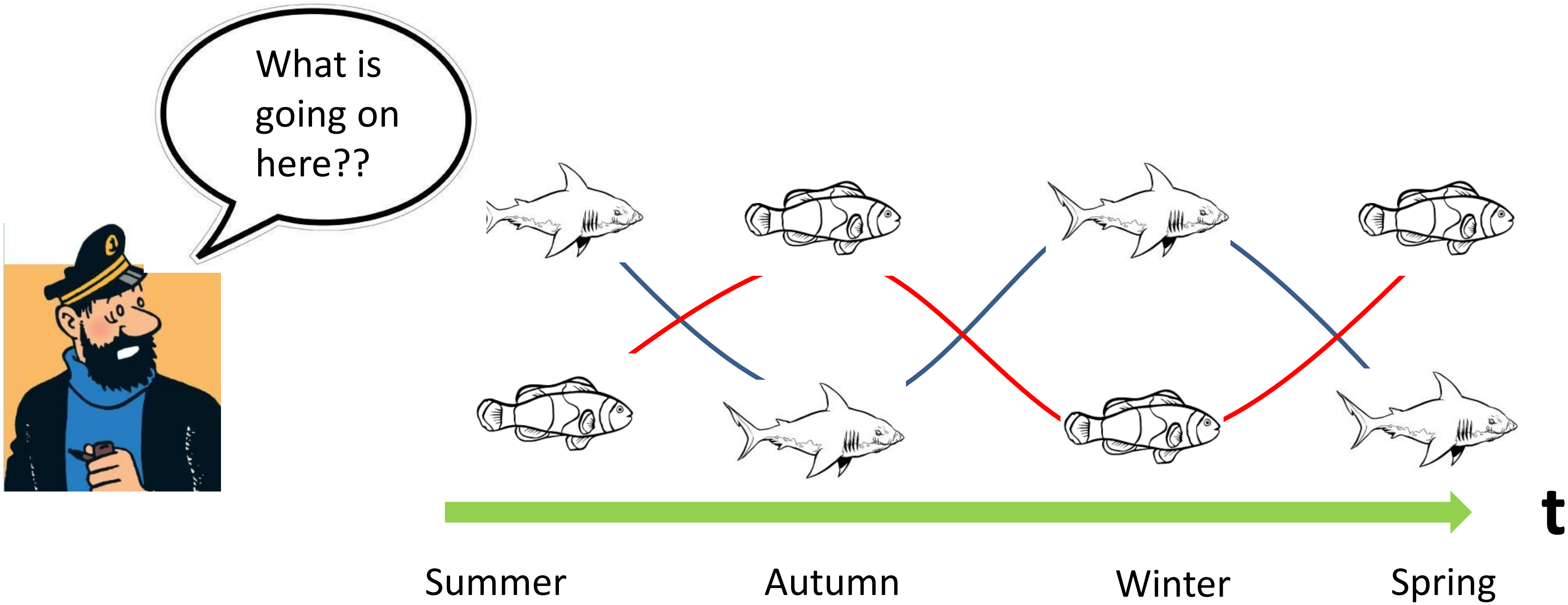
Lotka-Volterra System

Structure:

1. Historic origin & motivations
2. Assumptions and hypothesis of the Lotka-Volterra system
3. The mathematical model. Analytical solution of the Lotka-Volterra system (a little bit of differential equations)
4. Beyond the Lotka-Volterra system. More realistic models
5. Simulations
6. Conclussions and references

Lotka-Volterra System
1.Historic Origin & Motivations

Vito Volterra (1926) first proposed a model to explain the weird behaviour of the level of fish catches observed by the sailors in the Adriatic. Alfred J. Lotka developed the same model independently and simultaneously



There seemed to be a periodic behaviour on a two species population system !!

Lotka-Volterra System
2.Assumptions & Hypothesis

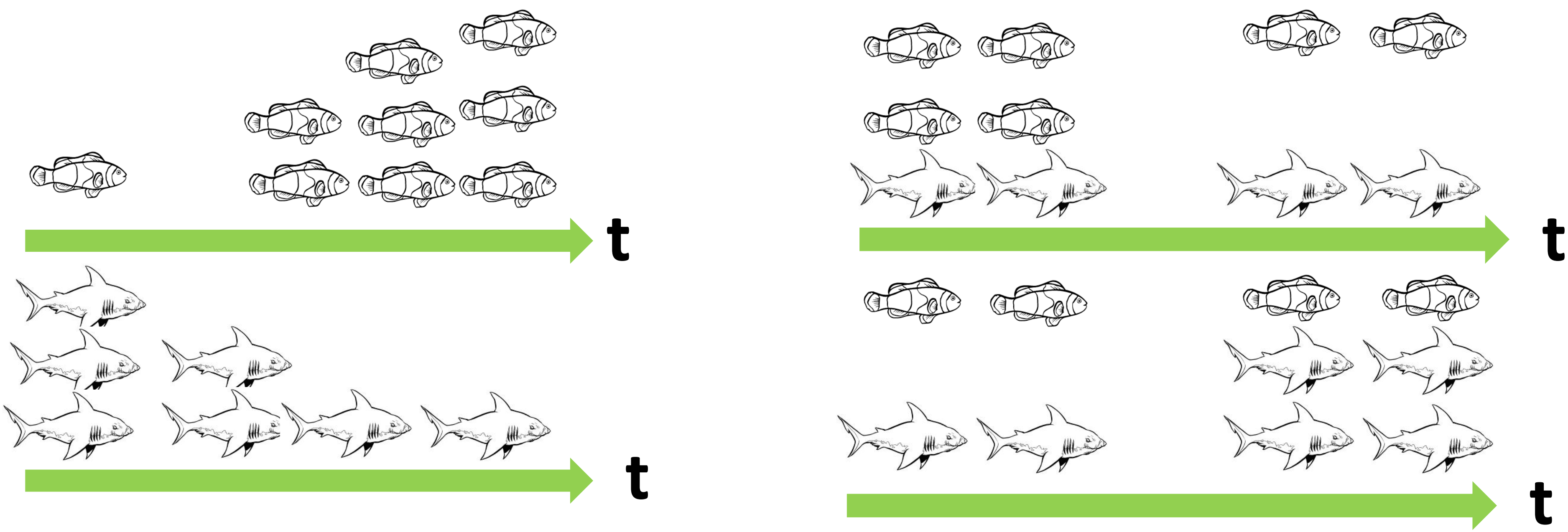
Volterra developed a model based in a few assumptions or observations:

1. The prey in abscense of predation grows unboundly.

2. In abscense of prey, the predator population decreases expontentially

3. The predator effect is to reduce the prey growth rate, proportional to both the predator and prey populations

4. The prey effect on the predator is to increase the predator growth rate somehow proportional to the prey and predator populations



Lotka-Volterra System
3.1. The mathematical model

The Lotka-Volterra mathematical model :

$$\begin{cases} \dfrac{dN}{dt} = N(a - bP) \\ \dfrac{dP}{dt} = P(cN - d) \end{cases} \quad (1)$$

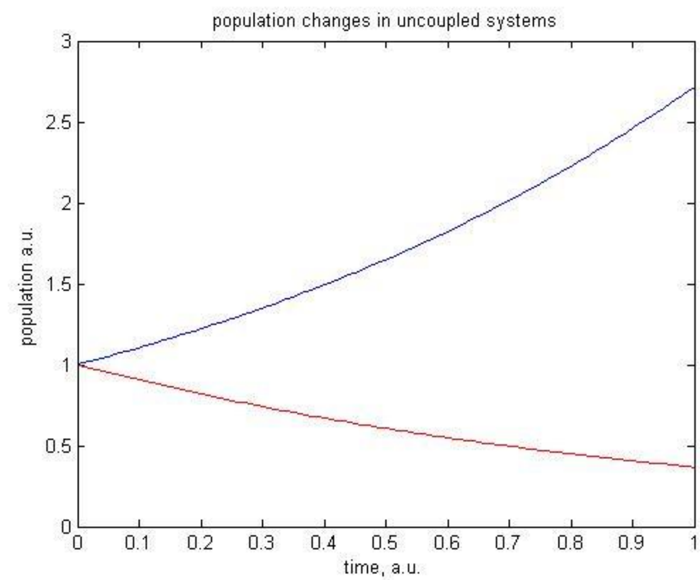
$N(t)$: *Number of preys*
$P(t)$: *Number of predators*
$a, b, c, d \rightarrow$ *constants*

Meaning of the constants:

a *and* ***d****, are the growth and decrease rate of isolated preys and predators respectively*

if ***b*** *and* ***c*** *are zero (uncoupled system)* $\Rightarrow$

$$\begin{cases} \dfrac{dN}{dt} = aN \\ \dfrac{dP}{dt} = -dP \end{cases}$$



Lotka-Volterra System
3.1. The mathematical model

$b \text{ and } c \text{ represent the coupling of the system}$:

$$\frac{dN}{dt} = N(a - bP)$$
$$\frac{dP}{dt} = P(cN - d)$$

$\boldsymbol{b}$ $represents\ the\ effect\ of\ predators\ in\ the\ prey\ population$

$\boldsymbol{c}$ $represents\ the\ effect\ of\ preys\ in\ the\ predator\ population$

$\boldsymbol{b}$ $is\ the\ only\ decreasing\ factor\ for\ the\ prey\ population$
$\rightarrow preys\ eaten\ by\ the\ predators$

$\boldsymbol{c}$ $is\ the\ only\ increasing\ factor\ for\ the\ predator\ popultion$
$\rightarrow a\ population\ growth\ \ proportional\ to\ the\ food\ available$

Lotka-Volterra System
3.2. Analytical Solution

First step: non-dimensionalize the system, through a change of variables:

$$u(\tau) = \frac{cN(t)}{d}, v(\tau) = \frac{bP(t)}{a}, \tau = at, \alpha = \frac{d}{a}$$

The system, then becomes:

$$\frac{du}{d\tau} = u(1 - v)$$

$$\frac{dv}{d\tau} = \alpha v(u - 1)$$

- This is a system of equations with only **one parameter**. α is the ratio between the decreasing and increasing factors of both predator and prey.
- It is an **autonomous system:** the independent variable do not appear in the right term in an explicit way.

Lotka-Volterra Systems
3.2. Analytical solution

Second step: Divide one by the other, to find the *trajectory equation:*

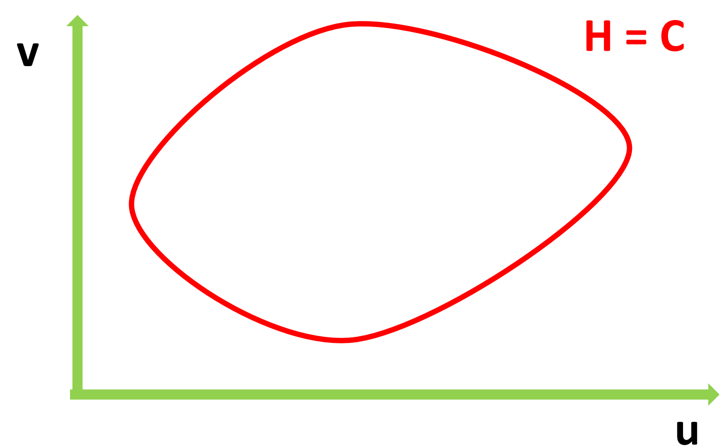
$$\frac{dv}{du} = \frac{\alpha v(u-1)}{u(1-v)} \quad (2)$$

This *trajectory equation* has a **first integral:**

$$\frac{d}{d\tau}\Big(H\big(u(\tau), v(\tau)\big)\Big) = 0 \;\forall\, \tau \quad \Longrightarrow \quad H\big(u(\tau), v(\tau)\big) = H\big(u(0), v(0)\big) = C \;\; \forall\, \tau$$

Which can be calculated , using the integrating factor $\left(\frac{1}{uv}\right)$ in the equation (2):

$$H(u, v) = \alpha u + v - \ln(u^{\alpha} v)$$



Result 1:
the existence of a first integral H, implies that the solution $(\boldsymbol{u}, \boldsymbol{v})$ belongs to the contour of H

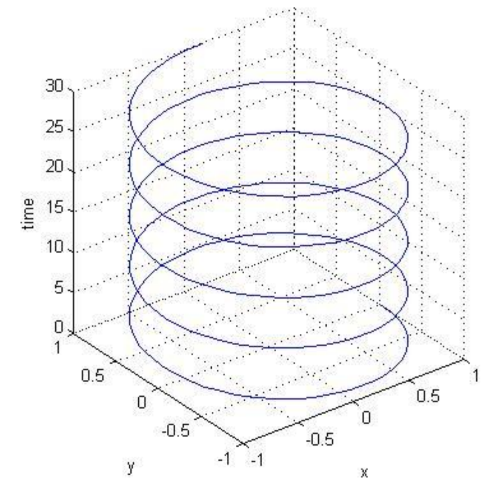
Lotka-Volterra System
3.2.Analytical solution

What is exactly a trajectory equation? What does "eliminate the time" means?

Example: harmonic oscillator: $\boldsymbol{\frac{d^2x}{dt^2} + \omega^2 x = 0}$

Whose solutions are:

$$x(t) = A\cos(\omega t + \theta)$$
$$\dot{x}(t) = -\omega A\sin(\omega t + \theta)$$



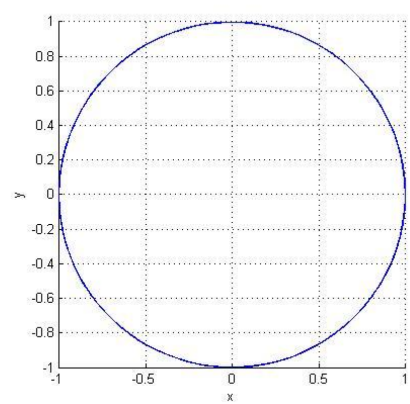
t

If we "eliminate the time" from the representation, we obtain the **trajectory equation:**

In this case $H(u,v) = H(x,\dot{x})$ is the Hamiltonian of the system:

$$H(x,\dot{x}) = \underbrace{\frac{1}{2}kx^2}_{\text{Potential energy}} + \underbrace{\frac{1}{2}m\dot{x}^2}_{\text{Kinetic Energy}}$$

→ Equation of a circle or elipse



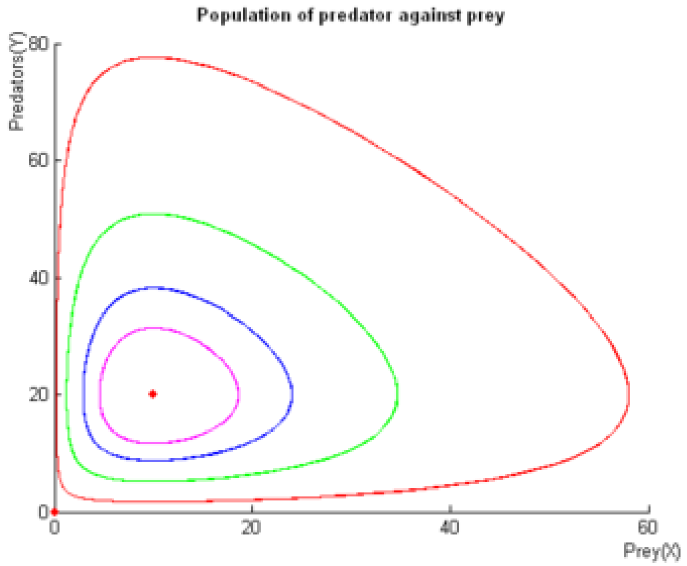
Phase space

Lotka-Volterra System
3.2. Analytical solution

Result 2. The different contour of H (trajectories) are closed curves → **The system has periodic solutions**

“Proof” (not rigorous):

- There is a fixed point in (1,1) $$\frac{du}{d\tau} = u(1-v); \frac{dv}{d\tau} = \alpha v(u-1)$$
- If we take polar coordinates around the minimum, H is an increasing function for every θ (direction).
- Neither u or v can approach to infinity (diverge) because it would cause H to diverge too

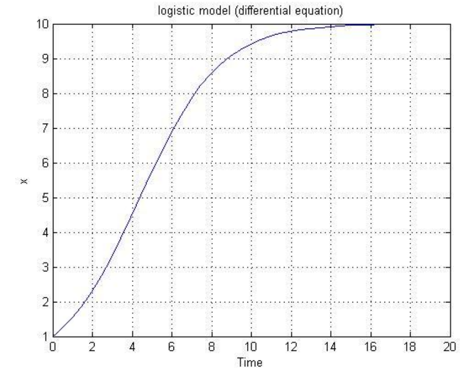


Lotka-Volterra System.
4. More realistic models

Problem: The classic Lotka-Volterra assumptions are very unrealistic. Specially the first one, where it is assumed that the prey population would increase indefinitely in absense of predators..

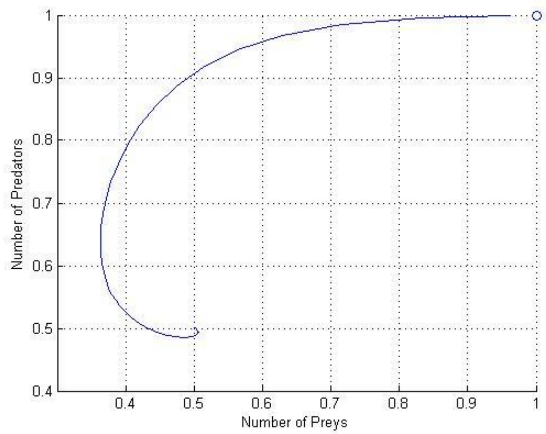
Logistic model:

$$\frac{dx}{dt} = rx(1 - \frac{x}{k})$$



One of the possible solutions (there are many approaches) is to assume logistic behavior in the prey and predator population:

$$\begin{cases} \dfrac{dN}{dt} = N(1 - N) - NP \\ \dfrac{dP}{dt} = P(1 - \dfrac{P}{N}) \end{cases}$$



New behavior !! There is an stable and attractive fixed point and the system no longer oscillates

Lotka-Volterra System
5. Simulations

Let's now see some simulations of Lotka-Volterra Systems

Lotka-Volterra System
6. Conclussions & References

Conclussions:

1. Lotka-Volterra equations forms an ideal model, but describes fairly well some experimental results and observations

2. The Lotka-Volterra System dynamics shows some very unique features:

> **2.1.** It present <u>stable oscillations</u> (except to the fixed point, and for zero initial values of either the prey or the predator)
>
> **2.2.** There is never extinction of neither the prey or predator if they were present in the initial conditions.

3. There are plenty of more reaslistic models with different behavior, usually without the oscillatory dynamics.

References:

J.D Murray. Mathematical Biology

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Gary William Flake. The Computational Beauty of Nature.

Thank you for your kind attention